Special Issue for The 5th Annual Conference on Theories and Applications of Basic and Biosciences ♦ September, 4th, 2021

Complete Regularity and Local Triviality of Fibrations

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submittion date: 29-06-2021 acceptance date: 13-07-2021 publishing date: 01-10-2021

Abstract:
In this paper, we relate completely regular maps to a version of fibration between Polish spaces. We present some characterizations of Polish fibrations, admitting a regular averaging operator. Furthermore, we weaken some conditions on completely regular maps in the Borel construction in our context. The findings of this paper generalize and extend some results in the literature.

Keywords: Fibrations, completely regular maps, Polish spaces.

Introduction

The study of completely regular maps appeared as an application to the theory of local triviality of fibrations (Dyer & Hamstrom 1958). The investigation of the problem of finding conditions which guarantee that a given map is a locally trivial fibration with a prescribed fibers has led to describing features of fibration theory which are related to the theory of continuous selections in the realm of Polish spaces (separable completely metrizable). A necessary condition to the mentioned problem is that all fibers to be homeomorphic, and in the case of metric spaces, that all fibers of close points to be homeomorphic under small transformations.
This article is devoted to completely regular maps (Dyer & Hamstrom 1958) and their application in the problem of fibrations between Polish spaces. Following some previous work including (Repovš et al. 1997; Ageev & Tymchatyn 2005; Dranishnikov 2016), the notions of completely regular maps, exact Milyutin maps, and Polish fibrations are shown to be related.
In the literature, the approach that links regular maps to locally trivial fibrations between Polish spaces is established in (Repovš et al. 1993, 1997) in a relation with the study of Milyutin maps. The historic development on the subject, in addition to an extensive collection of related results can be found in (Repovs & Semenov 2013). The classes of all (Milyutin) open maps between Polish spaces are characterized in (Ageev & Tymchatyn 2005). Valov (Valov 2009) presents some properties of Milyutin maps between Polish spaces. We further develop this idea by presenting some characterizations of Polish fibrations, admitting a regular averaging operator.
It is still unknown whether every completely regular map is a Hurewicz fibration (Dranishnikov 2016). However, it is proved in (Repovš et al. 1997) that a completely regular map between Polish spaces is locally trivial fibration, provided its point-inverses are homeomorphic to real line. Dranishnikov (Dranishnikov 2016) gives a complete discussion of completely regular maps in some constructions for compact metric spaces and compact fibers. This suggests a possible generalization to the case of Polish spaces where some restrictions may be weakened. We obtain a generalization where we drop the compactness condition.
Polish spaces form a large class of topological spaces, having various desirable properties and are quite manageable (Kechris 2012). In this article, by the fibrations of Polish spaces (or Polish fibrations) we mean the case where general Hurewicz fibra-

tions (Hurewicz 1955) of topological spaces are restricted to Polish spaces.

Preliminaries

In this section, we introduce some basic concepts and background related to Polish spaces and fibration theory that are required in this paper. The content of this section consists primarily of well-known definitions and facts from standard references. Further details can be found in the mentioned references.

1. Polish Spaces, Completely Regular Maps, and Spaces of Probability Measures (Bogachev 2007; Kechris 2012; Moschovakis 2009; Repovs & Semenov 2013)

A topological space X is *completely metrizable* if it admits a compatible metric, d such that (X, d) is complete. A separable completely metrizable space is called a *Polish space*. A *Polish group* is a topological group whose underlying space is Polish.

Let $g: X \to Y$. The space of all continuous $\mathbb{R}$-valued functions on X endowed with supremum norm is denoted by $C(X)$.

$$C(X) = \{f: X \to \mathbb{R} | f \quad is\ continuous\}$$

This space is a convex metric space. A correspondence

$$A: C(X) \to C(Y), \qquad \big(A(f)\big)(y) = f\big(g^{-1}(y)\big),$$

is called a *regular averaging operator*, if $A(f)$ is positive, the norm $||A|| = 1$, and $A(h\ o\ g) = h$ for $f \in C(X)$ and $h \in C(Y)$.

In this paper, every Polish space (X, d) is equipped with the Borel σ-algebra $\mathfrak{B}_X$ and a measure $\mu: X \to [0, \infty)$. A measure is called *a probability measure* if $\mu(X) = 1$. Let X be a Polish space. The space of all Borel probability measures on $X$is denoted by $P(X)$. We endow $P(X)$ with the weak-* topology. Equipped with the weak-* topology, the space $P(X)$ inherits many of the properties of X. Note that when X is compact and Polish, $P(X)$ becomes compact and Polish.

Definition 1 For a continuous map $f: X \to Y$, the corresponding mapping $P(f): P(X) \to P(Y)$ is defined by

$$(P(f)\mu)\phi = \int_X \phi \circ f d\mu, \quad \mu \in P(X) \quad and \quad \phi \in C(Y).$$

Moreover, for a continuous $f: X \to Y\ and\ g: Y \to Z$ we have $P(g \circ f) = P(g) \circ P(f)$ and $P(id_X) = id_{P(X)}$.

Definition 2 *Dirac measure* δ_x is a two-valued probability measure on a space X that is defined as

$$\delta_x(A) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

where $A \in \mathfrak{B}(X)$.

Definition 3 A map $f: (X, d) \to (Y, d')$ between metric spaces is said to be completely regular if for each $y \in Y$ and $\epsilon > 0$, there exists a $\delta > 0$ such that if $d'(y, y') < \delta$, then there is an ϵ-homeomorphism $\varphi: f^{-1}(y) \to f^{-1}(y')$, such that $d(x, \varphi(x)) < \epsilon$, for all $x \in f^{-1}(y)$.

Definition 4 Let X, Y be completely regular spaces. Then a continuous surjection $f: X \to Y$ is called a *Milyutin mapping* if there exists a continuous map $\nu: y \to P(X)$ such that $supp(\nu_y) \subset f^{-1}(y),\ y \in Y$.

2. Fibration Theory (Arkowitz 2011; Whitehead 1978).

Definition 5 Given a map $p: E \to B$. We say that p has the *homotopy lifting property*, denoted by HLP, if for every topological space X, every map $f: X \to E$, and every homotopy $H: X \times I \to B$ that begins with $p \circ f$ we can then lift H to a homotopy $\widetilde{H}: X \times I \to E$ that begins with f that is, such that

$$p \circ \widetilde{H} = H$$
$$\widetilde{H}(x, 0) = f(x).$$

If a map $p: E \to B$ has the HLP, then p is called a *(Hurewicz) fibration*. For a fibration, E is called the *total space* and B the *base space*. Given a point $b \in B$, then $F_b = p^{-1}(b)$ is the *fiber* of p at b. Thorough this paper, we will restrict the Hurewicz fibrations to Polish spaces and will be called *Polish fibrations*.

Definition 6 Let $p_i: E_i \to B_i \quad i = 1,2$ be two fibrations. A map $g: E_1 \to E_2$ is called *fiber-preserving* if it sends fibers into fibers, that is, if there exists a continuous $h: B_1 \to B_2$ such that the square

$$\begin{array}{ccc} E_2 & \xrightarrow{g} & E_1 \\ \downarrow{\scriptstyle p_2} & & \downarrow{\scriptstyle p_1} \\ B_2 & \xrightarrow[h]{} & B_1 \end{array}$$

commutes. Fibrations p_i are called *fiber-homotopy equivalent* if there are fiber-preserving maps $g: E_1 \to E_2$ and $g': E_2 \to E_1$ and fiber-preserving homotopies between gg' and 1_{E_1}, and between $g'g$ and 1_{E_2} .We write $A \times_g E$ for the *pullback* of a given fibration $p: E \to B$ by a map $g: A \to B$.

$$\begin{array}{ccc} A \times_g E & \xrightarrow{g'} & E \\ \downarrow{\scriptstyle q} & & \downarrow{\scriptstyle p} \\ A & \xrightarrow[g]{} & B \end{array}$$

A fibration $p: E \to B$ with fiber F is *fiber-homotopy trivial* if it is fiber-homotopy equivalent to the projection map $pr_B: B \times F \to B$. A fibration is *locally trivial* if there is a covering $\{U_\lambda\}$ of the base B, such that the restrictions $p: p^{-1}(U_\lambda) \to U_\lambda$ are fiber-homotopy trivial. In addition, when p is a locally trivial fibration and B is connected, then all fibers of p are homeomorphic.

Definition 7 *7* Let X be a topological space and G a group. A continuous *(right) action* of G on X is a continuous map $\lambda: X \times G \to X$, $\lambda(x, g) = x \cdot g$ (denoted $(x, g) \mapsto x \cdot g$) such that $x \cdot 1 = x$ for all $x \in X$ and for all $g \in G$. The *orbit* of $x \in X$ is the set of points $x \cdot g$ for all $g \in G$. The *quotient* X/G is the set of all orbits of the group action on X, and the map $: X \to X/G$ sending $x \to X$ to its orbit is called the *orbit map*.

Definition 8 The action is *free* if for each $x \in X$ the subgroup $\{g \in G | x \cdot g = x\}$ is the trivial subgroup $\{e\}$. The action is *proper* if the action map λ is proper. Recall that a map is called proper if the inverse images of compact sets are compact.

Main Results

Proposition 1 Let $f: X \to Y$ be a completely regular map between compact Polish spaces. Let K be a closed subset of Y. If f admits a regular averaging operator, then the restriction

$$f|_{f^{-1}(K)}: f^{-1}(K) \to K$$

admits a regular averaging operator.
Proof. Applying the space of probability measures functor P to f *and* $f|_{f^{-1}(K)}$ yields,

$$P(f): P(X) \to P(Y), \tag{1}$$

and

$$P(f|_{f^{-1}(K)}): P(f^{-1}(K)) \to P(K). \tag{2}$$

Since f is completely regular and admits a regular averaging operator, then by (Argyros & Arvanitakis 2002, Proposition 6), there exists an injection

$$g: P(Y) \to P(X)$$

such that

$$P(f) \circ g = id_{P(Y)}.$$

According to (Banakh 1995, Theorem 2.14), the functor P preserves inverse images, that is,

$$P(f)^{-1}(P(K)) = P(f^{-1}(K)).$$

Thus,

$$\begin{aligned} g(P(K)) &\subset P(f)^{-1}(P(K)) \\ &\subset P(f^{-1}(K)). \end{aligned} \tag{3}$$

Therefore,

$$g|_{P(K)}: P(K) \to P(f^{-1}(K))$$

is a choice function of $f|_{f^{-1}(K)}$. *This implies that* $f|_{f^{-1}(K)}$ admits a regular averaging operator. Q.E.D.

Proposition 2 Let $f: X \to Y$ be an exact Milyutin map between Polish spaces with $\nu: Y \to P(X)$. Let $K \neq \emptyset$ be an open subset of X. Then the set

$$W = \{\mu \in \nu(Y) \mid f^{-1}(y) \cap K \neq \emptyset\}$$

is dense in $P(X)$.

Proof. From the definition of the exact Milyutin map, and since $\mu \in P(X)$, we have $supp(\mu_y) = f^{-1}(y)$. Thus; the result follows immediately from the fact that the set $\{\mu \in P(X) \mid supp(\mu) \cap K \neq \emptyset\}$ is dense in $P(X)$ (Valov 2009, Lemma4.1). Q.E.D.

Theorem 1 Let X and Y be compact Polish spaces, and let $f: X \to Y$ be a completely regular map. Then f admits a regular averaging operator if and only if there exists a Polish subspace $K \subset X$ such that $f|_K: K \to Y$ is a homeomorphism.

Proof. Recall that when f is completely regular, all fibers $f^{-1}(y)$ of close points are homeomorphic under small transformations. First, assume that f admits a regular averaging operator. Then, there exists a continuous map $\nu: Y \to P(X)$ such that ν is supported by $f^{-1}(y)$ (Argyros & Arvanitakis 2002, Proposition 6). Since the spaces are compact and Polish, then

$$supp(\nu_y) = f^{-1}(y), \quad y \in Y. \tag{4}$$

Define a continuous map $\varphi: Y \to X$ such that

$$f \circ \alpha = id_Y, \tag{5}$$

and

$$\mu_y = \delta_{\alpha(y)}. \tag{6}$$

Then, $\varphi(Y) \subset X$ is closed and

$$f|_{\varphi(y)}: \varphi(y) \to Y$$

is a homeomorphism. From Equation (4), f is an exact Milyutin map. Thus, by (Ageev & Tymchatyn 2005), there exists a Polish subspace $K \subset X$ such that $f|_K: K \to Y$ is an open surjection with

$$supp(\nu_y) \cap K = (f|_K)^{-1}(y), \quad y \in Y. \tag{7}$$

Thus, for all $y \in Y$, we obtain that

$$\begin{aligned} \emptyset \neq (f|_K)^{-1}(y) &= f^{-1}(y) \cap K \\ &= supp(\nu_y) \cap K \\ &\subseteq \{\varphi(y)\}. \end{aligned} \tag{8}$$

Therefore, $K = \varphi(Y)$.

On the other hand, suppose that there exists a closed set $K \subset X$ such that

$$f|_K : K \to Y$$

is a homeomorphism. Let

$$\varphi = (f|_K)^{-1} : Y \to K.$$

Then, for $\nu : Y \to P(X)$ defined by $\nu_y = \delta_{\varphi(y)},\ y \in Y$, Equation (4) holds. Let

$$K = \cup_{y \in Y}\ supp(\nu_y).$$

Thus, for all $y \in Y$

$$\begin{aligned} supp(\nu_y) &= f^{-1}(y) \cap K \\ &= (f|_K)^{-1}(y) \\ &= \{\varphi(y)\}. \end{aligned} \tag{9}$$

Therefore, f is an exact Milyutin map, and consequently, f admits a regular averaging operator. Q.E.D.

Example 1 Let E be the space $E = S^1 \times [0,1]/\sim$ where $\sim$ is the equivalence relation $S^1 \times [0,1]$ generated by $(z,t) \sim (-z,-t)$, $z \in S^1$ and $t \in [0,1]$. Her, we view the circle S^1 as a subset of $\mathbb{C}$. Define the fibration $f: E \to S^1$ such that $f(z,t) = z$. The map f takes $S^1 \times \{0\} \subset E$, bijectively onto S^1. Let $g: S^1 \to S^1 \times \{0\})$, $g(z) = (z,0)$ be the continuous inverse map to the restriction $f|_{S^1 \times \{0\}}$. Then, the map $f|_{S^1 \times \{0\}}: S^1 \times \{0\} \to S^1$ is a homeomorphism. On the other hand, E and S^1 are compact metric spaces and f is an open surjection, then there exists a continuous linear onto map $u: C(E) \to C(S^1)$ defined by $(u(h))(z) = h(f^{-1}(z))$, satisfying the condition:

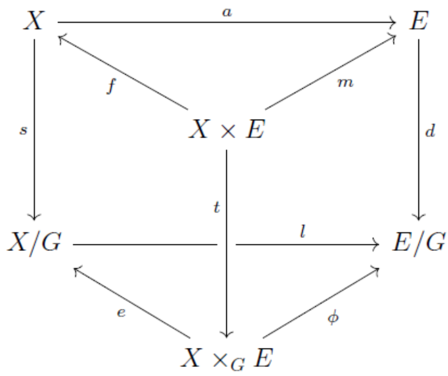
$$inf(h(f^{-1}(z))) \leq (u(h))(z) \leq sup(h(f^{-1}(z)))$$

for every $h \in C(E)$ and $z \in S^1$ (Michael 1964). Hence, u is continuous with norm 1. Moreover, u is a regular averaging operator since for any $\phi \in C(S^1)$,

$$u(\phi \circ f)(z) = \phi \circ f(f^{-1}(z)) = \phi(z)$$

Thus, f is a continuous open surjection, which admit a regular averaging operator u.

Next, let a group G act freely on spaces X and E with the projections s and d onto orbit spaces, then there is a commutative diagram called the Borel construction (Borel 1960) as shown below.



It was shown in (Dranishnikov 2016) that if G is a compact group, E is a metric space, and X is a compact space, then ϕ is completely regular. We obtain the following generalization where we drop the compactness condition.

Theorem 2 Let G be a Polish group which acts freely and properly on Polish spaces X and E. Then the projection $\phi: X \times_G E \to E/G$ in the Borel construction is completely regular provided, E/G is locally path connected and ϕ is a proper map.

Proof. Consider maps as defined above in Borel construction diagram. Let E/G be locally path connected, and let ϕ be a proper map. Since the action of G is free and proper, then ϕ is a fibration. Therefore, according to (Addis 1972), this implies that for each $e_0 \in E/G$ and $\epsilon > 0$, there is a $\delta > 0$ such that the metric $\rho(e_0, e) < \delta$ implies that there are small maps

$$\alpha: \phi^{-1}(e) \to \phi^{-1}(e_0)$$
$$\beta: \phi^{-1}(e_0) \to \phi^{-1}(e)$$

and small homotopies

$$\alpha \circ \beta \simeq id$$
$$\beta \circ \alpha \simeq id$$

in the respective fibers. Let $H_t = \beta \circ \alpha$. Since α is a small map,

$$\alpha \circ H_t: \phi^{-1}(e) \to \phi^{-1}(e_0)$$

is a small homotopy. Thus, if $\epsilon > 0$ is given, then there is a $\delta > 0$ such that $\rho(e_0, e) < \delta$ implies that there is a small equivalence

$$\alpha: \phi^{-1}(e) \to \phi^{-1}(e_0)$$

for which $\rho(\alpha(x), x) < \epsilon$. This yields complete regularity. Q.E.D.

Example 2 Let $S^1 = \{z \in \mathbb{C} | |z| = 1$ and let $p: S^1 \to S^1/\mathbb{Z}_2$ be the orbit map where the group of integers modulo 2, $\mathbb{Z}_2$, acts on S^1 by $z \mapsto -z$. This action is free and proper and p is a locally trivial fibration and hence, p is an open map. Let $S^1 \times S^1$ be the product of two circles. Define an action of $\mathbb{Z}_2$ on $S^1 \times S^1$ by $(z, w) \mapsto (\frac{1}{z}, -w)$. The result is the quotient space $S^1 \times_{\mathbb{Z}_2} S^1$. By Borel construction we have,

$$\phi: S^1 \times_{\mathbb{Z}_2} S^1 \to S^1/\mathbb{Z}_2$$

such that $\phi([z, z']) = [p(z)]$ for every $z, z' \in S^1$. The fiber of ϕ is S^1. Since the Polish space $S^1 \times S^1$ is continuously mapped onto $S^1 \times_{\mathbb{Z}_2} S^1$, then $S^1 \times_{\mathbb{Z}_2} S^1$ is a Polish space. Fix $y \in p^{-1}(x)$. Since p is open, any sequence x_n converging to x in $S^1/\mathbb{Z}_2$ admits a lift y_n converging to y in S^1 with respect to orbit map p. Then, we can find a sequence of homeomorphisms between the fibers $\phi^{-1}(x)$ and $\phi^{-1}(x_n)$ in

$S^1 \times_{\mathbb{Z}_2} S^1$ that converges to the identity map between $\phi^{-1}(x)$ and $\phi^{-1}(x)$. Therefore, ϕ is completely regular.

Theorem 3 Let the spaces spaces X, E and G, and the map ϕ be as defined in Theorem 2. Then the following hold:

1. If X is homeomorphic to the real line, then ϕ is a locally trivial fibration;
2. If X is perfect (i.e., without isolated points) with respect to ϕ, then ϕ is a Milyutin map.

Proof. Since X, E and G are all Polish spaces, Then $X \times_G E$ is a Polish space. According to (Repovš et al. 1997), every completely regular map between Polish spaces with fibers homeomorphic to $\mathbb{R}$ is a locally trivial fibrations. The fiber of ϕ is X, and since ϕ is completely regular, then ϕ is a locally trivial fibration. Thus, part 1 of Theorem 3 is proved. On the other hand, a result of (Valov 2009) shows that every open surjection between Polish spaces with perfect fibers is a Milyutin map. By the first part of Theorem 3, ϕ is a locally trivial fibration which implies that ϕ is an open surjection. This proves part 2. Q.E.D.

Conclusion

In this article, the notion of fibration for Polish spaces are examined by establishing a direction that combines effectively, the study of the descriptive set theory, measure theory and fibration theory. The results show that the notions of completely regular maps, exact Milyutin maps, and Polish fibrations can be linked in such a way that further extends the concepts involved.

REFERENCES

1) Addis, D.F. (1972) A strong regularity condition on mappings. General Topology and its Applications 2(3): 199–213.
2) Ageev, S. & Tymchatyn, E. (2005) On exact atomless milutin maps. Topology and its Applications 153(2-3): 227–238.
3) Argyros, S.A. & Arvanitakis, A.D. (2002) A characterization of regular averaging operators and its consequences. Studia Mathematica 151: 207–226.
4) Arkowitz, M. (2011) Introduction to homotopy theory. New York: Springer Science & Business Media.
5) Banakh, T. (1995) The topology of spaces of probability measures, I: functors $p\tau$ and bP. Matematychni Studii 5(1-2): 65–87.
6) Bogachev, V.I. (2007) Measure theory. Berlin: Springer Science & Business Media.
7) Borel, A. (1960) Seminar on Transformation Groups.(AM-46). New Jersey: Princeton University Press.
8) Dranishnikov, A.N. (2016) On some problems related to the Hilbert-Smith conjecture. Sbornik: Mathematics 207(11): 1562.
9) Dyer, E. & Hamstrom, M.E. (1958) Completely regular mappings. Fundamenta Mathematicae 45(1): 103–118
10) Hurewicz,W. (1955) On the concept of fiber space. Proceedings of the National Academy of Sciences 41(11): 956–961.
11) Kechris, A. (2012) Classical descriptive set theory. New York: Springer Science &Business Media
12) Michael, E. (1964) A linear mapping between function spaces. Proceedings of the American Mathematical Society 15(3): 407-409.

13) Moschovakis, Y. (2009) Descriptive Set Theory. Mathematical surveys and monographs. Providence RI: American Mathematical Society.
14) Repovš, D., Semenov, P.V. & Šcepin, E.V. (1993) On zero-dimensional milutin maps and michael selection theorems. Topology and its Applications 54(1-3): 77–83.
15) Repovš, D., Semenov, P.V. & Šcepin, E.V. (1997) On exact milyutin mappings. Topology and its Applications 81(3): 197–205.
16) Repovs, D. & Semenov, P.V. (2013) Continuous selections of multivalued mappings. Dordrecht: Springer Science & Business Media.
17) Valov, V. (2009) Probability measures and milyutin maps between metric spaces. Journal of Mathematical Analysis and Applications 350(2): 723–730..
18) Whitehead, G. (1978) Elements of Homotopy Theory. Graduate texts in mathematics. New York: Springer-Verlag.

الانتظام الكلي والبساطة المحلية لدوال الفايبريشن

الملخص:

في هذه الورقة البحثية، نقوم بربط مفهوم الدوال المنتظمة الكلية مع نوع خاص من دوال الفايبريشن بين الفضاءات الطوبولوجية البولندية. أيضا نقدم بعض خصائص لدوال الفايبريشن بين الفضاءات الطوبولوجية البولندية التي تمتلك عامل توسيط منتظم. بالإضافة الى ذلك، فإننا نخفف بعض الشروط المتعلقة ببناءات Borel في الدوال المنتظمة الكلية. النتائج التي توصلت إليها هذه الورقة هي تعميم وتوسيع لبعض النتائج في بحوث سابقة.

الكلمات المفتاحية: دوال فايبريشن، دوال منتظمة كليا، فضاءات بولندية.